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# A lower bound for the volume of Dirichlet fundamental polyhedrons(Complex Analysis on Hyperbolic 3-Manifolds)

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CITATION:

Furusawa, Harushi. A lower bound for the volume of Dirichlet fundamental polyhedrons(Complex Analysis on Hyperbolic 3-Manifolds). 数理解析研究所講究録 1994, 882: 11-25

ISSUE DATE:

1994-08

URL:

<http://hdl.handle.net/2433/84249>

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# A lower bound for the volume of Dirichlet fundamental polyhedrons

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1. Introduction. This paper is concerned with a lower bound for the volume of Dirichlet fundamental polyhedrons for Kleinian groups.

Let  $H^3 = \{(x, y, t) \in \mathbb{R}^3; t > 0\}$  with metric  $d(\cdot, \cdot)$  induced by the line element  $ds^2 = (dx^2 + dy^2 + dt^2)/t^2$ . Let  $f$  be an element of  $PSL(2, \mathbb{C})$  and identifies with a Möbius transformation of  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  onto itself, such as

$$f(z) = \frac{az+b}{cz+d}, \quad ad-bc=1.$$

Its action on the Riemann sphere  $\hat{\mathbb{C}}$  can be naturally extended to  $H^3$ .

Next for each  $f$  and  $g$  in  $PSL(2, \mathbb{C})$ , we let  $[f, g]$  denote the commutator  $fgf^{-1}g^{-1}$ . We define the two complex numbers

$$(1) \quad \beta(f) = \text{tr}^2(f) - 4, \quad \gamma(f, g) = \text{tr}([f, g]) - 2,$$

for the two generator subgroup  $\langle f, g \rangle$ .

The following inequality [4] gives an important necessary condition for a two generator group  $\langle f, g \rangle$  to be nonelementary and discrete.

Proposition 1([J1]). If  $\langle f, g \rangle$  is nonelementary and discrete, then

$$(2) \quad |\gamma(f, g)| + |\beta(f)| \geq 1 \quad \text{and} \quad |\gamma(f, g)| + |\beta(g)| \geq 1.$$

The above inequality is called Jørgensen's inequality.

Proposition 2([R]). If  $\langle f, g \rangle$  is a nonelementary discrete Fuchsian subgroup of  $\text{PSL}(2, \mathbb{R})$ , then

$$(3) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

On the other hand, if  $\langle f, g \rangle$  is not Fuchsian, it is known that there is a nonelementary discrete group  $\langle f, g \rangle$  with an arbitrary small number  $|\gamma(f, g)|$  ([J2]). But if we have some restrictive conditions for elements of generators, one can obtain the lower bounds for  $|\gamma(f, g)|$ . If we have a restriction  $\beta(f) = \beta(g)$  for two-generator group  $\langle f, g \rangle$ . Jørgensen proved in [J3] that

$$(4) \quad |\gamma(f, g)| \geq 1/8.$$

The first purpose of this paper is to show the existence of a collar from the consideration of a lower bound of  $|\gamma(f, g)|$ . This gives an improvement of a previous paper [2].

2 Preliminary. We collect some elementary results.

Lemma 1. If  $f$  and  $g$  are in  $\text{PSL}(2, \mathbb{C})$  with  $\gamma(f, g) = \gamma$  and  $\beta(f) = \beta$ , then

$$(5) \quad \gamma(f, gfg^{-1}) = \gamma(\gamma - \beta) \text{ and } \beta([f, g]) = \gamma(\gamma + 4).$$

Lemma 2. Let  $G$  be an elementary discrete subgroup of  $\text{PSL}(2, \mathbb{C})$ . If

$f, g \in G$  with  $\gamma(f, g) \neq 0$ ,  $\beta(f) = \beta(g) \neq -4$ , then

$$(6) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/5) (=0.3819\cdots).$$

It is easily seen that  $\gamma(f,g) \neq 0$  if and only if  $\text{fix}(f) \cap \text{fix}(g) = \emptyset$  where  $\text{fix}(f)$  denotes the fixed point set in  $\hat{C}$ . Now we need to show the followings.

Lemma 3. If  $\langle f,g \rangle$  is discrete with  $\gamma(f,g) = \beta(f) \neq 0$ , then either  $f$  is elliptic of order 2,3,4, or 6 or  $g$  is elliptic of order 2.

3. A lower bound for the commutator. We show here a lower bound for the commutator of two generator groups.

Next we show Theorem 1 which is applicable to the collar lemma.

Theorem 1. Let  $\langle f,g \rangle$  be a discrete subgroup of  $\text{PSL}(2,C)$  with  $\gamma(f,g) \neq 0$ ,

$\beta(g) \neq -4$  and

(7)  $0 < |\beta(f)| < 2\{2\cos(2\pi/7) - 2\cos(\pi/7) + 1\}$ , then

$$|\gamma(f,g)| > 2 - 2\cos(\pi/7).$$

Proof. Put  $\gamma = \gamma(f,g)$  and  $\beta = \beta(f)$ . Suppose  $|\gamma(f,g)| \leq 2 - 2\cos(\pi/7)$ .

If  $\gamma = \beta$ , then Lemma 3 yields  $|\gamma| = |\beta| \geq 1$  or  $\beta(g) = -4$  which contradict the assumption of Theorem 1. Therefore  $\gamma \neq \beta$  and we have that  $\langle [f,g], f[f,g]f^{-1} \rangle$

is nonelementary discrete subgroup of  $\langle f,g \rangle$  by Lemma 2 with

$$0 < |\gamma([f,g], f[f,g]f^{-1})| = |\gamma^2(\gamma - \beta)(\beta + 4)| < 0.3 \text{ and } |\beta([f,g])| = |\gamma(\gamma + 4)| < 1.$$

We have the following result from Jørgensen's inequality (2) that

$$(8) \quad 1 \leq |\gamma^2(\gamma - \beta)(\beta + 4)| + |\gamma(\gamma + 4)|.$$

Specially  $|\beta| > |\gamma|$ , if not we have

$$1 \leq |\gamma^2(\gamma - \beta)(\beta + 4)| + |\gamma(\gamma + 4)| \leq (2|\gamma^3| + |\gamma|)(|\gamma| + 4) < 1.$$

Set  $S = \{z; |z| \leq 2 - 2\cos(\pi/7)\}$ . Let  $R$  be the union of the convex hulls of  $S \cup \{-4\}$  and  $S \cup \{\beta\}$ . The function  $u(z) = |z+4| + |z-\beta|$  is subharmonic in

$D = \text{int}(R)$  and hence there exists a point  $\xi$  in  $\partial D$  such that

$$|\gamma + 4| + |\gamma - \beta| \leq u(\xi). \text{ Put } 0 \leq \theta = |\arg \beta| \leq \pi, \text{ then we have follows by}$$

estimating  $u(\xi)$  from the above by the length of the component of  $\partial D \setminus \{-4, \beta\}$

which contains  $\xi$ , that is

$$(9) \quad |\gamma + 4| + |\gamma - \beta| \leq r_1 + r_2 + |\gamma|\theta,$$

where  $r_1 = (4^2 - |\gamma|^2)^{1/2} + |\gamma| \sin^{-1}(|\gamma|/4)$  and

$r_2 = (|\beta|^2 - |\gamma|^2)^{1/2} + |\gamma| \sin^{-1}(|\gamma|/|\beta|)$ . Set  $r = r_1 + r_2$  where

$4.91 < r < 4.918$ . Then (12) is reformed by

$$(10) \quad 1 \leq |\beta + 4|(\theta - 1)|\gamma|^3 + \{|\beta + 4|(r - 4) + 1\}|\gamma|^2 + 4|\gamma|.$$

Our goal is to estimate that the right hand side of above inequality is less than 1 and this contradicts the assumption  $|\gamma| \leq 2 - 2\cos(\pi/7) = d$ .

Now  $|\beta + 4|^2 = |\beta|^2 + 4^2 + 8|\beta|\cos\theta = a + b\cos\theta$  where  $16.7 \leq a \leq 16.8$  and

$7.1 \leq b \leq 7.2$ . Set  $F(x) = (a + b\cos\theta)^{1/2}(\theta - 1)x^3 + \{(a + b\cos\theta)^{1/2}(r - 4) + 1\}x^2 + 4x$  where

$0 \leq x \leq d$ . The derivative  $F'(x)$  is a increasing function with respect to  $x$ ,

then  $F(x) \leq F(d)$  where  $d = 2 - 2\cos(\pi/7)$ . Let  $f(\theta) = F(d)$  and therefore we have

$f'(\theta) = (a + b\cos\theta)^{-1/2}(d^2/2)g(\theta)$  where

$$g(\theta) = 2(a + b \cos \theta)d - b\{(\theta - 1)d + (r - 4)\} \sin \theta.$$

We divide  $\theta$  into 13 cases;

(I) If  $0 \leq \theta \leq 4\pi/9$ , then  $g(\theta) > 0$  and  $f(\theta) \leq f(4\pi/9) \leq 0.997789 < 1$ .

(II) If  $4\pi/9 \leq \theta \leq 5\pi/11$ , then we have

$$f(\theta) \leq \{16.8 + 7.2 \cos(4\pi/9)\}^{1/2} (5\pi/11 - 1)d^3 + \\ [\{16.8 + 7.2 \cos(4\pi/9)\}^{1/2} (r - 4) + 1]d^2 + 4d < 0.99884 < 1.$$

(III) If  $5\pi/11 \leq \theta \leq 8\pi/17$ , similarly we have  $f(\theta) \leq 0.999444 < 1$ .

(IV) If  $8\pi/17 \leq \theta \leq 9\pi/19$ , similarly we hold  $f(\theta) \leq 0.998055 < 1$ .

(V) If  $9\pi/19 \leq \theta \leq \pi/2$ , similarly  $f(\theta) \leq 0.99785 < 1$ .

(VI) If  $10\pi/21 \leq \theta \leq 11\pi/23$ , then  $f(\theta) \leq 0.99785 < 1$

(VII) If  $11\pi/23 \leq \theta \leq \pi/2$ , also we have  $f(\theta) \leq 0.999901$

(VIII) If  $\pi/2 \leq \theta \leq 5\pi/9$ , then  $g(\theta) < -0.64$  and  $f(\theta) \leq f(\pi/2) < 0.9974 < 1$ .

(IX) If  $5\pi/9 \leq \theta \leq 2\pi/3$ , also we have  $g(\theta) < -0.47$  and  $f(\theta) < 1$

(X) If  $2\pi/3 \leq \theta \leq 11\pi/15$ , we have  $f(\theta) \leq 0.99936 < 1$ .

(XI) If  $11\pi/15 \leq \theta \leq 7\pi/9$ , then we have  $f(\theta) \leq 0.995193 < 1$ .

(XII) If  $7\pi/9 \leq \theta \leq 8\pi/9$ , also we have  $f(\theta) \leq 0.9995 < 1$ .

(XIII) If  $8\pi/9 \leq \theta \leq \pi$ , also  $f(\theta) \leq 0.9986 < 1$ .

This completes the proof.

Theorem 2([G&M2]). Let  $\langle f, g \rangle$  be a discrete subgroup of  $\text{PSL}(2, \mathbb{C})$  with

$\gamma(f, g) \neq 0$ ,  $\beta(f) = \beta(g) \neq -4$  then

$$(11) \quad |\gamma(f, g)| \geq 0.193.$$

The above constant is not sharp. The following theorem is sharp.

Theorem 3([G&M2]). Let  $\langle f, g \rangle$  be a discrete subgroup of  $\text{PSL}(2, \mathbb{C})$  with

$$\gamma(f, g) \neq 0, \quad \beta(f) = \beta(g) \neq -4 \text{ and}$$

$$(12) \quad \min\{|\beta(f)|, |\beta(fg)|, |\beta(fg^{-1})|\} \geq 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$|\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

The following two theorems are also proved in [F2].

Theorem 4([F2]). Let  $\langle f, g \rangle$  be a nonelementary discrete group with

$$\beta(g) \neq -4 \text{ and } 0 < |\beta(f)| < 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$(13) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7) \text{ or } |\gamma(f, g) - \beta(f)| > 1.$$

Theorem 5([F2]). Let  $\langle f, g \rangle$  be a nonelementary discrete group with

$$\beta(g) \neq -4 \text{ and } 0 < |\beta(f)| < 2\{\cos(2\pi/7) + 2\cos(\pi/7) - 1\}, \text{ then}$$

$$(14) \quad \max\{|\gamma(f, g)|, |\gamma(f, gfg^{-1})|\} \geq 2 - 2\cos(\pi/7).$$

4. The collar lemma. Let  $G$  be a discrete subgroup of  $\text{PSL}(2, \mathbb{C})$  acting on the

upper half space  $H^3$ . If  $f \in G \setminus \{\text{id.}\}$  is not a parabolic element, then we denote  $A_f$  the geodesic in  $H^3$  joining the fixed points of  $f$  on  $\hat{C}$  the boundary of  $H^3$  in

$\mathbb{R}^3$ . For a positive number  $k$ , we define a tubular neighborhood about  $A_f$  as

$$N_k(f) = \{x \in H^3; d(x, A_f) \leq k\},$$

where  $d$  is the hyperbolic metric. Let  $G_f$  be the subgroup of  $G$  which leaves  $A_f$  invariant. We call  $N_k(f)$  a collar for  $f$  in  $G$ , if  $g(N_k(f)) \cap N_k(f) = \emptyset$  for all  $g \in G \setminus G_f$  and  $g(N_k(f)) = N_k(f)$  for all  $g \in G_f$ . The number  $k$  is called the width of the collar  $N_k(f)$ .

Following [F1], we introduce the notion of complex distance between two geodesics in  $H^3$  and also state the cosine rule. Denote a directed geodesic  $L$  by the ordered pair of its endpoints; so  $L=(a,b)$  for its endpoints  $a,b \in \mathbb{C}$ ,  $a \neq b$ . The complex distance  $t = \delta(L_1, L_2) \in \mathbb{C}$  between two directed geodesics  $L_1=(a_1, b_1)$  and  $L_2=(a_2, b_2)$  is defined as follows:  $|\operatorname{Re}(t)| \geq 0$  is the hyperbolic distance between the geodesics and  $\operatorname{Im}(t)$  is the angle made by the geodesics along their common perpendicular and is determined modulo  $2\pi$  unless  $\operatorname{Re}(t) \neq 0$ , in which case  $\pm \operatorname{Im}(t)$  is determined modulo  $2\pi$ . We can compute the complex distance by the formula ([F1]),

$$(15) \quad \cosh^2(t/2) = (a_1, a_2, b_2, b_1).$$

The right hand side of this equality denotes the cross ratio of these four points. Therefore, for any  $f \in \operatorname{PSL}(2, \mathbb{C})$ , we see  $\delta(L_1, L_2) = \delta(f(L_1), f(L_2))$ .

Let  $f \in \operatorname{PSL}(2, \mathbb{C})$  be non-parabolic and let  $A_f$  be directed geodesics in the hyperbolic space joining the fixed points of  $f$ . If  $L$  is a perpendicular to  $A_f$  then the complex distance  $t$  between  $L$  and  $f(L)$  is called the complex



translation length of  $f$ . In this case, we have

$$(16) \quad \text{tr}^2(f) = 4\cosh^2(t/2),$$

which makes sense even if  $f$  is not loxodromic.

For the geodesics  $L_0, L_1, L_2$ , put  $\omega = \delta(L_1, L_2)$ ,  $t_1 = \delta(L_0, L_1)$ ,  $t_2 = \delta(L_0, L_2)$  and denote by  $\alpha$  the complex distance from the perpendicular between  $L_0$  and  $L_1$  to the perpendicular between  $L_0, L_2$ . Then we have the so-called cosine rule:

$$(17) \quad \cosh(\omega) = \cosh(t_1)\cosh(t_2) - \cosh(\alpha)\sinh(t_1)\sinh(t_2).$$

Let  $\omega$  be the complex distance between  $A_f$  and  $A_{gfg^{-1}}$ . Then we can normalize  $f$  and  $gfg^{-1}$  as follows:

$$f = \begin{bmatrix} \cosh(t/2) & \sinh(t/2) \\ \sinh(t/2) & \cosh(t/2) \end{bmatrix}, gfg^{-1} = \begin{bmatrix} \cosh(t/2) & \exp(\omega)\sinh(t/2) \\ \exp(-\omega)\sinh(t/2) & \cosh(t/2) \end{bmatrix}$$

We have  $\gamma = \text{tr}(fgf^{-1}g^{-1}) - 2 = -(1 - \cosh(t))(1 - \cosh(\omega))$ . Recall the cosine rule

(17) and take  $L_2 = A_f$ ,  $L_0 = A_g$  and  $L_1 = A_{gfg^{-1}} = g(A_f)$ . It is easy to show that

$$\mu = \delta(A_g, A_f) = \delta(A_g, A_{gfg^{-1}}). \text{ Thus we have } \cosh(\omega) = \cosh^2(\mu) - \cosh(t')\sinh^2(\mu)$$

where  $t'$  is a complex translation length of  $g$ . Therefore we have the following

lemma([F1], [K]).

Lemma 4. Let  $f$  and  $g$  be non-parabolic elements in  $\text{PSL}(2, \mathbb{C})$  and let  $\mu$  be the complex distance between  $A_f$  and  $A_g$ . If  $\gamma = \text{tr}(fgf^{-1}g^{-1}) - 2 \neq 0$ , then

$$(18) \quad 4\gamma = \beta(f)\beta(g)\sinh^2(\mu).$$

Making use of Lemma 4 and Theorem 1, we will show the following so-called collar lemma.

Theorem 6. Let  $G$  be a nonelementary discrete subgroup of  $\text{PSL}(2, \mathbb{C})$ . Let  $f$  be an element of  $G \setminus \{\text{id.}\}$  with  $0 < |\beta(f)| = 2s < 2\{2\cos(2\pi/7) - 2\cos(\pi/7) + 1\} = 2c$ , then there exist a collar  $N_{k(s)}(f)$  with the width

$$(19) \quad \sinh^2 k(s) = (c/s - 1)/2.$$

And further let  $f$  and  $g$  be in  $G$  and suppose that  $f$  and  $g$  generate a nonelementary discrete group. If  $0 < |\beta(f)| = 2s < 2c$  and  $0 < |\beta(g)| = 2s' < 2c$ , then the collars  $N_{k(s)}(f)$  for  $f$  and  $N_{k(s')}(g)$  for  $g$  are disjoint, where  $k$  is the function defined by (19).

Proof. Let  $f$  be an element of  $G \setminus \{\text{id.}\}$  with  $0 < |\beta(f)| < 2c$  and  $g \in G \setminus G_f$ .

Suppose  $f$  is elliptic. The condition  $|\beta(f)| < 1$  implies that the order of  $f$  is not less than 7. Then  $\langle f, gfg^{-1} \rangle$  is not elementary discrete subgroup of  $G$ . If  $f$  is not elliptic, then we see that  $\mu \neq 0$  where  $\mu$  is the complex distance between  $A_f$  and  $A_{gfg^{-1}}$  for  $g \in G \setminus G_f$ . Thus we conclude that  $\langle f, gfg^{-1} \rangle$  is non-elementary discrete group and we have from (2),  $|\beta(f)| + |\gamma(f, gfg^{-1})| \geq 1$ .

Therefore we have  $\gamma(f, gfg^{-1}) \neq 0$  by the assumption of Theorem 7. Thus

$\langle f, gfg^{-1} \rangle$  is discrete with  $\gamma(f, gfg^{-1}) \neq 0$ ,  $\beta(f) = \beta(gfg^{-1}) \neq -4$ , then we have

$|\gamma(f, gfg^{-1})| \geq 2 - 2\cos(\pi/7) = c^2$  for any  $g \in G \setminus G_f$  by Theorem 1. It is already

known that  $4\gamma(f, gfg^{-1}) = \beta^2(f) \sinh^2(\mu)$  from (18). By the simple calculation

we have  $c = |\beta(f)| |\sinh(\mu)| \leq |\beta(f)| \{2\sinh^2(\operatorname{Re} \mu / 2) + 1\}$ . This completes the

first part of theorem.

Next we prove the last part of theorem. Let  $\mu$  be the complex distance between  $A_f$  and  $A_g$ . Then (18) and (19) imply

$$\begin{aligned} |\sinh^2(\mu)| &= 4|\gamma(f, g)| / (|\beta(f)| |\beta(g)|) \\ &\geq c^2 / (|\beta(f)| |\beta(g)|) \\ &= (2\sinh^2 k(s) + 1)(2\sinh^2 k(s') + 1) \\ &= (\cosh^2 k(s) + \sinh^2 k(s))(\cosh^2 k(s') + \sinh^2 k(s')) \\ &\geq \{\cosh k(s) \cosh k(s') + \sinh k(s) \sinh k(s')\}^2 \\ &= \cosh^2(k(s) + k(s')), \end{aligned}$$

where  $k$  is the function defined by (19). From  $\cosh^2 \operatorname{Re} \mu \geq |\sinh^2 \mu|$ , we have

$\operatorname{Re} \mu \geq k(s) + k(s')$ , which proves the last part of the Theorem. We complete

the proof.

Remark. The function  $k(s)$  defined on the above is decreasing function with respect to  $s$  and  $k(s) \rightarrow \infty$  as  $s \rightarrow \infty$ .

Let  $f$  be a loxodromic element in  $\text{PSL}(2, \mathbb{C})$  with the multiplier  $\exp(2\alpha(f) + 2i\theta)$ . The translation length  $2\alpha(f)$  of  $f$  is also defined by  $\inf\{d(\xi, f(\xi)); \xi \in \mathbb{H}^3\}$ . Next lemma is given by Zagier ([Me2]).

**Lemma 5.** Let  $x_1, x_2 \in \mathbb{R}$  and  $0 < x_1 < \pi/\sqrt{3}$ , then there exist a positive integer  $n$  such that

$$(20) \quad \cosh nx_1 - \cos nx_2 \leq \cosh(\sqrt{4\pi x_1/\sqrt{3}}) - 1.$$

If the multiplier of  $f$  is given by  $\exp(2\alpha(f) + 2i\theta)$ , then

$$(21) \quad |\beta(f)| = 2(\cosh 2\alpha(f) - \cos 2\theta) \\ = 4(\sinh^2 \alpha(f) + \sin^2 \theta).$$

If  $f$  is a loxodromic, then the axes of  $f$  and  $f^n$  ( $n \neq 0$ ) are same. By a simple

computation and lemma 5 have  $|\beta(f^n)| = 2(\cosh 2n\alpha(f) - \cos 2n\theta) \leq$

$2\{\cosh(\sqrt{8\pi\alpha(f)/\sqrt{3}}) - 1\}$  for some positive integer  $n$ . We restate Theorem 6,

setting  $2\alpha(f) = L$ ,  $\cosh(\sqrt{4\pi L/\sqrt{3}}) - 1 = s < c_1 = 2\cos(2\pi/7) - 2\cos(\pi/7) + 1 (< 0.445)$

and  $\frac{\sqrt{3}}{4\pi} [\log(1 + c_1 + \sqrt{c_1^2 + 2c_1})]^2 = c_2 (= 0.114519)$ , then

**Theorem 7.** Let  $g$  be a non trivial closed geodesic with the length  $L(g) < c_2$  in any complete hyperbolic 3-manifold  $M$ , then there exists a tubular neighbourhood  $N(g)$  around  $g$  in  $M$ . Let  $r$  be the hyperbolic width of  $N(g)$ . Then, the hyperbolic volume of  $N(g)$  is  $\pi \cdot L(g) \cdot \sinh^2 r$  which is a decreasing function of  $L(g)$ .

Remark. If  $L(g) \leq 0.10857$ ,  $r \geq 0.17198$  and  $\pi \cdot L(g) \cdot \sinh^2 r \geq 0.01018$ .

5. A lower bound of the volume of  $V(H^3/\Gamma)$ . Let  $q(z_1, z_2)$  be a chordal distance between  $z_1, z_2 \in \hat{\mathbb{C}}$ , that is  $q(z_1, z_2) = 2|z_1 - z_2|(1 + |z_1|^2)^{-1/2}(1 + |z_2|^2)^{-1/2}$ .

We introduce two different norms which measure the distance from  $f$  in mobius transformation groups to  $\text{id}$ . The first of the two norms for  $f$  is given in terms of the matrix,

$$(22) \quad m(f) = ||f - f^{-1}||$$

where for any matrix  $A$  in  $SL(2, \mathbb{C})$  we let  $||A||$  denote its euclidean norm

$||A||^2 = \text{tr}(AA^*)$  and  $A^*$  its Hermitian transpose. The second is defined by

$$(23) \quad \rho(f) = d(f(j), j)$$

where  $j$  is the point  $(0, 0, 1)$  in  $H^3$ .

If  $f$  is in  $\hat{M}(\mathbb{C})$  with  $\text{fix}(f) = \{z_1, z_2\}$ , then  $|\beta(f)| = \frac{1}{2} \frac{q(z_1, z_2)^2}{8 - q(z_1, z_2)^2} m(f)^2$

,  $2\cosh(\rho(f)) = ||f||^2$  and simple computation leads  $4||f||^2 = m(f)^2 + 2|\text{tr}^2(f)|$ .

If  $f$  is in  $\hat{M}(\mathbb{C}) \setminus \{\text{id}\}$  with  $\text{fix}(f) = \{z_1, z_2\}$  and multiplier  $\exp(2\alpha(f) + 2i\theta)$ ,

then,

$$(24) \quad \sinh^2(\rho(f)/2) = \frac{4 - q^2}{8 - q^2} \frac{m(f)^2}{8} + \sinh^2 \alpha(f) ,$$

$$(25) \quad \sinh^2(\rho(f)/2) = \frac{4}{8 - q^2} \frac{m(f)^2}{8} - \sin^2 \theta ,$$

where  $q = q(z_1, z_2)$ .

Let  $N$  denote the set of positive integers and for each  $\rho$  in  $[0, \infty)$  set

$$(26) \quad s(\alpha) = \sup_{\theta} \left( \inf_N (\sinh^2 k \alpha + \sin^2 k \theta) \right).$$

Then  $s(x)$  is nonnegative, nondecreasing and continuous in  $[0, \infty)$  with  $s(0)=0$ .

Moreover from a lemma due to Zagier [Me2], it follows that

$$(27) \quad s(t) \leq \sinh^2 \sqrt{a} t, \quad a = 2\pi / \sqrt{3},$$

for  $0 \leq t < \sqrt{3}\pi/2$ . Then we have the following lemma.

Lemma 7 ([G&M2]). Suppose that  $a$  and  $c$  are positive constants, and  $f$  is in  $M(\mathbb{C}) \setminus \{\text{id.}\}$  with two distinct fixed points and multiplier

$\exp(2\alpha(f) + 2i\theta)$ . If  $m(f^k)^2 \geq c$  for any  $k$  in  $N$  and

$$(28) \quad s(a) \leq c/8 + \sinh^2 \alpha(f) - \sinh^2 a,$$

then  $\rho(f) \geq 2a$ .

The following is an immediate result from Jørgensen's inequality.

Lemma 8. Let  $\langle f, g \rangle$  be a non-elementary discrete group, then

$$(29) \quad m(f)m(g) \geq 4(\sqrt{2}-1).$$

If  $m(f)m(g) < 4(\sqrt{2}-1)$  for a discrete group  $\langle f, g \rangle$ , then  $\langle f, g \rangle$  is a elementary

group. Next lemma gives more simple result for generators if each generative

elements have the following restrictive conditions.

Lemma 9. Let  $\langle f, g \rangle$  be a discrete group with

$m(f)^2 < 4(\sqrt{2}-1)$  and  $m(g)^2 \leq 4(\sqrt{2}-1)$ , then  $\text{fix}(f) = \text{fix}(g)$ .

The following proposition is an important role in this paper which is due to P. Waterman[W].

Proposition 4. Let  $G$  be a discrete group. Then there is a mobius transformation  $h$  for  $G$  satisfying

$$(30) \quad m(f) \geq 4(\sqrt{2}-1)$$

for any element  $f$  in  $hGh^{-1} \setminus \{id.\}$ .

Theorem 8. Let  $M$  be a complete hyperbolic 3-manifold, then

$$V(M) > 0.001.$$

Proof. It is known that  $M$  is represented by  $H^3/\Gamma$  for a torsion free Kleinian group  $\Gamma$ . If  $\Gamma$  contains parabolic transformations, then  $V(M) \geq \sqrt{3}/4$  ([Me1]).

From now on, we consider that  $\Gamma$  is a purely loxodromic transformations group.

we set  $2\alpha' = \inf[2\alpha(f); f \in \Gamma \setminus \{id.\}]$  where  $\exp(2\alpha(f) + 2i\theta)$  is a multiplier of  $f$ . If  $2\alpha' \leq 0.10857$ , then Theorem 7 and Remark shows that  $M$  has a collar

with the volume not less than 0.001. Otherwise, we consider a conjugate group

for  $\Gamma$  stated in Proposition 4 and reset this conjugate group  $\Gamma$ , then we have

$m(f)^2 \geq 4(\sqrt{2}-1)$  for  $f$  in  $\Gamma \setminus \{id.\}$ . Now we seek  $\eta$  satisfying the equation

$s(\eta) = (\sqrt{2}-1)/2 + \sinh^2 \alpha' - \sinh^2 \eta$ . we have  $2\eta \geq 0.11576$ . Thus

$\rho(f) \geq 0.11576$ , thus the Dirichlet fundamental polyhedron centered at  $j$  for  $\Gamma$

contains a ball  $B$  with hyperbolic radius 0.11576. Then we have the result.

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